Also solved by Arkady Alt, San Jose, CA; Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; David Diminnie, Texas Instruments, Incorporated, Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica,Campus de Bellaterra, Barcelona, Spain; Boris Rays, Brooklyn NY, and the proposer.

- **5241:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $\alpha \geq 0$ be a real number. Calculate

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria , Spain

For $x \in [0,1]$, $\alpha \leq x^n + \alpha$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{\alpha}\ dx\right)^n &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n \\
\alpha &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.
\end{aligned}$$

On the other hand, since function $y = x^n$ is convex for $n \geq 1$, by Jensen's inequality

$$\begin{aligned}
\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n &\leq \int_0^1 (x^n+\alpha)\ dx \\
\lim_{n\to\infty}\int_0^1 x^n+\alpha\ dx &\leq \lim_{n\to\infty}\frac{1}{n+1}+\alpha = \alpha.
\end{aligned}$$

So, $\displaystyle\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n = \alpha.$

Solution 2 by Arkady Alt, San Jose, CA

Let $\displaystyle a_n = \int_0^1 \sqrt[n]{x^n+\alpha}dx$. Note that $\displaystyle\lim_{n\to\infty} a_n = 1$.

Indeed, we have

$\displaystyle\sqrt[n]{\alpha} = \int_0^1 \sqrt[n]{\alpha}dx \leq a_n \leq \int_0^1 \sqrt[n]{1+\alpha}dx = \sqrt[n]{1+\alpha}$ and $\displaystyle\lim_{n\to\infty}\sqrt[n]{\alpha} = \lim_{n\to\infty}\sqrt[n]{1+\alpha} = 1.$

Since $\displaystyle\lim_{n\to\infty} a_n^n = \lim_{n\to\infty} e^{n\ln a_n}$ we will find $\displaystyle\lim_{n\to\infty} n\ln a_n$.

Since

$$\begin{aligned}
\lim_{n\to\infty}(a_n-1) &= 0 \text{ we have} \\
\lim_{n\to\infty} n\ln a_n &= \lim_{n\to\infty} n\ln(1+(a_n-1)) \\
&= \lim_{n\to\infty}\left(n(a_n-1)\cdot\frac{\ln(1+(a_n-1))}{(a_n-1)}\right)
\end{aligned}$$

$$= \lim_{n\to\infty} n\,(a_n - 1) \text{ because}$$
$$\lim_{n\to\infty} \frac{\ln(1+(a_n-1))}{(a_n-1)} = 1.$$

Thus, it suffices to find $\lim_{n\to\infty} n\,(a_n - 1)$.

Since

$$\begin{aligned}
n\,(a_n - 1) &= n\left(\int_0^1 \sqrt[n]{x^n+\alpha}dx - 1\right) \\
&= n\int_0^1 \left(\left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) + \left(\sqrt[n]{\alpha}-1\right)\right)dx \\
&= n\left(\sqrt[n]{\alpha}-1\right) + n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right)dx \text{ and} \\
\lim_{n\to\infty} n\left(\sqrt[n]{\alpha}-1\right) &= \lim_{n\to\infty} n\left(e^{\ln\alpha n} - 1\right) \\
&= \ln\alpha
\end{aligned}$$

then it remains to find

$$\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right)dx.$$

By the Mean Value Theorem

$$\frac{\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}}{x^n} = \frac{1}{n\sqrt[n]{\theta^{n-1}}} \text{ where } \theta \in (\alpha, x^n+\alpha).$$

Hence,

$$\frac{\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}}{x^n} < \frac{1}{n\sqrt[n]{\alpha^{n-1}}} \iff \sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha} < \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}$$

and, therefore,

$$0 < n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right)dx < n\int_0^1 \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}dx = \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}}.$$

Since

$$\lim_{n\to\infty} \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}} = 0,$$

by the Squeeze Principle,

$$\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right)dx = 0.$$

Thus,

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}dx\right)^n = e^{\ln\alpha} = \alpha.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

1. For $a = 0$ the limit is trivially $0 = a$.
2. For $a > 0$. We set $I_n^n = \left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)^n = \exp\left(n\ln\left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)\right) = e^{A_n}$.

   Now, considering that $n \in [1,+\infty)$, since $0 < \sqrt[n]{x^n+a} \le 1+a$ and

   $\sqrt[n]{x^n+a} \overset{n\to+\infty}{\longrightarrow} 1$ for $x \in [0,1]$, by dominated convergence theorem we get that

   $I_n \to 1$, thus $\ln I_n \to 0$.

   Furthermore, by Leibniz's rule we have that for $n \ge 1$

$$\frac{\partial I_n}{\partial n} = \int_0^1 \frac{\partial}{\partial n}\sqrt[n]{x^n+a}\,dx = \int_0^1 (x^n+a)^{\frac{1-n}{n}}\left(\frac{nx^n\ln x - (x^n+a)\ln(x^n+a)}{n^2}\right)dx.$$

   We also have that

$$\begin{aligned}\left|(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a) - nx^n\ln x\right)\right| &\le \frac{1+a}{a}\left(|(x^n+a)\ln(x^n+a)| + |nx^n\ln x|\right.\\ &\le \frac{1+a}{a}\left(\max\{e^{-1},(1+a)\ln(1+a)\} + e^{-1}\right)\end{aligned}$$

   and since

$$(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a) - nx^n\ln x\right) \to \begin{cases}\ln(1+a), & \text{if } x = 1\\ \ln a, & \text{if } x \in [0,1)\end{cases}$$

   by the dominated convergence theorem it is $-n^2\dfrac{\partial I_n}{\partial n} \to \ln a$.

   Now applying De l' Hospital's rule we get

$$\lim_{n\to+\infty} A_n = \lim_{n\to+\infty}\frac{\ln I_n}{n^{-1}} = \lim_{R\ni n\to+\infty}\frac{\ln I_n}{n^{-1}} \overset{0/0}{=} \lim_{n\to+\infty} I_n^{-1}\cdot\left(-n^2\frac{\partial I_n}{\partial n}\right) \to \ln a,$$

so the required limit in each case is a.

Solution 4 by Adrian Narco, Polytechnic University, Tirana, Albania

The function, $f(x) = \sqrt[n]{x^n+\alpha} = (x^n+\alpha)^{\frac{1}{n}}$, is strictly increasing and everywhere continuous on $[0;1]$, thus we can apply the mean value theorem for integral, that is,

$$\exists c \in (0;1) : \int_0^1 \sqrt[n]{x^n+\alpha}\mathrm{dx} \quad = \quad f(c)(1-0) = (c^n+\alpha)^{\frac{1}{n}}$$